PRICING RISKS WHEN STANDARD DEVIATION PRINCIPLE IS APPLIED FOR THE PORTFOLIO

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ABSTRACT

When the risk loading for the whole portfolio is set proportionally to the standard deviation, then the problem of coherent pricing of individual risks arises. A similar problem was considered by Borch (1962), who proposed to make use of the equilibrium solution for the *n*-person game. However, the solution is suited only for small *n*, rather reflecting the game played by few companies negotiating the merger of their portfolios. Otto (2004) proposed the "intuitively appealing" approximation that leads to a simple pricing formula for the case of large *n*. The paper presents the proof that the approximation is justified.

Key words: premium principles, top-down approach, n-person game, convergence in distribution

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1. INTRODUCTION

It is widely accepted that the insurance premium should be based on the top-down approach, entailing two (or more) steps. The first step, based on risk and return considerations on the whole company level, should lead to the premium formula for the whole portfolio to be decomposed next for premiums for individual risks. Sometimes intermediate steps (pricing groups of risks) are needed, when the capital backing risk is to be allocated between separately managed regional divisions or lines of business.

The problem arises when the pricing criteria applied on the company level lead to non-additive premium formulas. In particular, under a number of different sets of assumptions the pricing formula with the safety loading proportional to the standard deviation is obtained. Usually it is due to the assumption that the aggregate amount of claims over a year is approximately normal, and the safety criterion is based on the one-year possible loss. However, also the probability of ruin in the long run, when decisions on premium and the desired amount of initial capital are undertaken jointly, leads to the loading proportional to the standard deviation (as in Buhlmann 1985).

2. MARGINAL PREMIUM FORMULA AND THE BALANCING PROBLEM

Let us assume that the premium formula for the whole portfolio takes a form:

$$(1) \qquad \Pi(W) = \mu_W + \alpha\sigma_W ,$$

where $W = X_1 + X_2 + ... + X_n$ is the aggregate amount of claims for the whole portfolio that consists of n individual risks $X_1, X_2, ..., X_n$, and μ_W, σ_W are expectation and standard deviation of the portfolio. The indifference price of the additional risk X is given by the marginal premium formula:

$$(2) \qquad \Pi_{m\arg}(X) \doteq \Pi(W + X) - \Pi(W) = \mu_X + \alpha(\sigma_{W+X} - \sigma_W)$$

Assuming now that standard deviations of the portfolio with and without the additional risk do not differ too much we can make an approximation:

$$\sigma_{W+X} - \sigma_W = (\sigma_{W+X} - \sigma_W)\frac{\sigma_{W+X} + \sigma_W}{\sigma_{W+X} + \sigma_W} = \frac{\sigma_{W+X}^2 - \sigma_W^2}{\sigma_{W+X} + \sigma_W} \approx \frac{2\operatorname{cov}(X,W) + \sigma_X^2}{2\sigma_W},$$

that leads to the marginal premium formula:

$$(3) \qquad \Pi_{m\arg}(X) = \mu_X + \frac{\alpha}{\sigma_W}\left(\operatorname{cov}(X,W) + \frac{1}{2}\sigma_X^2\right).$$

In case of the risk X_i already included in the portfolio W the covariance $\operatorname{cov}(X_i, W)$ contains already the variance component, so the marginal premium formula reads:

$$\Pi_{m\arg}(X_i) = \mu_i + \alpha\sigma_W^{-1}\left(\operatorname{cov}(X_i,W) - \frac{1}{2}\sigma_i^2\right)$$

However, the sum of marginal premiums calculated for all risks from the portfolio does not suffice to cover the aggregate risk:

$$\Pi(W) - \sum_{i=1}^{n}\Pi_{m\arg}(X_i) = \frac{\alpha}{2\sigma_W}\sum_{i=1}^{n}\sigma_i^2 .$$

The balancing problem concerns only this part of the loading that is due to variances, whereas the covariance part is balanced. Assuming additionally zero covariances we come to the result:

$$\Pi(W) - \sum_{i=1}^{n}\Pi_{m\arg}(X_i) = \frac{\alpha}{2}\sigma_W$$

showing that in this case the deficit equals exactly one half of the required loading for the portfolio. As including covariances is irrelevant to the balancing problem, all the remaining part of the paper concerns the case of uncorrelated risks.

The balancing problem is nothing strange – the excess of average cost over the marginal cost is a common effect of returns to scale, and in a way it stands for the essence of the diversification effect obtained by composing large portfolios of risks by an insurance company. However, the problem of allocating the missing half of the loading $\alpha\sigma_W$ remains. An intuitively appealing rule is to allocate the "common safety fund" proportionally to the individual contribution to the loading. This leads to the basic pricing formula:

$$\Pi_b(X) = \mu_X + \alpha \frac{\sigma_X^2}{\sigma_W}. \tag{4}$$

However, the following alternative solution is also balanced:

$$\Pi_b(X) = \mu_X + \alpha\sigma_W \left(\frac{\sigma_X^2}{2\sigma_W} + \frac{c_{X,k}}{2c_{W,k}} \right),$$

despite it allocates the missing half of the safety loading proportionally to cumulants of order *k*, not necessarily equal 2. Also, any linear combination of variants (for various *k*) of the above formula is balanced, and we need some criteria to choose between them.

3. JUSTIFICATION OF THE BASIC PREMIUM FORMULA (4)

In order to explain the solution proposed by Borch (1962), let us imagine that all risks are insured sequentially. Under the particular ordering of risks from the basic portfolio $\{X_1, \ldots, X_j, X_{j+1}, \ldots X_n\}$ the risk X_{j+1} is priced as if the first *j* risks were already insured. Thus the corresponding marginal premium formula reads:

$$\Pi_{marg}(X_{j+1}) = \mu_{j+1} + \alpha\left(\sqrt{\sigma_1^2 + \ldots + \sigma_{j+1}^2} - \sqrt{\sigma_1^2 + \ldots + \sigma_j^2}\right)$$

Obviously the sum of premiums calculated this way for all risks in the portfolio is equal to the portfolio premium, but the loading is allocated very unequally, charging heavily those risks that have been insured earlier, and only slightly those that have been insured the latest. Borch suggested setting the price of a given risk as an average of prices calculated according to this rule under all *n*! possible orderings, showing that this rule is just equivalent to the equilibrium solution of the *n*-person game (derived first by Shapley in 1953). Borch has recommended this solution as suited to the problem of allocating the loading between few groups of risks rather than to large number of individual risks. Indeed, one can hardly imagine calculations for large *n*. This would require averaging over 2^{n-1} possibly different values, as that large is the number of subsets of the set of remaining $(n-1)$ risks, when one of *n* risks is priced.

However, for the case of large *n* the approximate solution can be derived. Let us consider again the problem of pricing an additional risk *X* when each of its possible $(n+1)$ positions in the sequence $\{X_1, \ldots, X_j, X, X_{j+1}, \ldots X_n\}$ is treated as equally probable, and all *n*! orderings of remaining risks are equally probable, too. Let us denote the set of basic risks preceding the risk *X* in a particular ordering by *PRE*. The derivation is based on the remark that under certain conditions the ratio of the sum of variances of preceding risks (elements of *PRE*) in the total variance of all risks in the portfolio is approximately uniformly distributed over the unit interval:

(5) $$\underset{u\in[0,1]}{\forall} \Pr\left(\sum_{j\in PRE} \sigma_j^2 \le u \sum_{j=1}^{n} \sigma_j^2 \right) \approx u \,.$$

Let us make use of this approximation now, leaving its justification to the next section. According to the approximation we can write:

$$\mathrm{E}\left(\sqrt{\sum_{i\in PRE} \sigma_i^2 + \sigma_X^2} - \sqrt{\sum_{i\in PRE} \sigma_i^2} \right) \approx \int_0^1 \left(\sqrt{u\sigma_W^2 + \sigma_X^2} - \sqrt{u\sigma_W^2} \right) du \,.$$

Using the abbreviated notation $c \doteq \sigma_X^2 / \sigma_W^2$ we can express the above integral as:

$$\int_0^1 \left(\sqrt{u\sigma_W^2 + \sigma_X^2} - \sqrt{u\sigma_W^2} \right) du = \sigma_W \int_0^1 \left(\sqrt{u+c} - \sqrt{u} \right) du = \frac{2}{3}\sigma_W \left[(1+c)^{3/2} - 1 - c^{3/2} \right].$$

Taking now into account that c is small and so $\sqrt{1+c} \approx 1 + c/2$ we come to the result:

$$\int_0^1 \left(\sqrt{u\sigma_W^2 + \sigma_X^2} - \sqrt{u\sigma_W^2} \right) du \approx \frac{2}{3}\sigma_W \left[(1+c)\left(1 + \frac{c}{2}\right) - 1 - c^{3/2} \right],$$

that is a polynomial in $c^{1/2}$ with the greatest term of order c. Omitting smaller terms yields:

$$\mathrm{E}\left(\sqrt{\sum_{i\in PRE} \sigma_i^2 + \sigma_X^2} - \sqrt{\sum_{i\in PRE} \sigma_i^2} \right) \approx \sigma_W c = \frac{\sigma_X^2}{\sigma_W} \,,$$

which indeed leads to the balanced pricing formula (4).□

The above derivation was proposed by the author in the book written in Polish (2004) and in English (2005, Chapter 20, co-author Paweł Miśta). However, in both cases the approximation (5) has been justified “intuitively”. Sections 4 and 5 contains the rigorous justification in terms of the theorem on convergence in distribution and its refinements.

4. THEOREM ON CONVERGENCE IN DISTRIBUTION

Approximation (5) can be justified by the convergence of the distribution of the share of risks preceding a given risk in the variance of the portfolio to the uniform distribution. The convergence requires that under increasing size of the portfolio the share of maximal variance in the variance of the whole portfolio vanishes:

(6) $$\lim_{n\to\infty} \left(\max\{y_1, y_2, \ldots, y_n\} \right) = 0 \,, \quad \text{where:} \quad y_j \doteq \sigma_j^2 / \left(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 \right)$$

In order to prove the sufficiency of the above requirement, some more precise definitions, notations and assumptions are needed. All of them are summarized below:

(A) $E = \{e_1, e_2, \ldots, e_n\}$ is a basic set of elements,

(B) $y : E \to [0,1]$ is a function that assigns the real nonnegative number $y_j \doteq y(e_j)$ to each element of the basic set, such that $y_1 + y_2 + \ldots + y_n = 1$

(C) $M = \max_{j=1,\ldots,n} \{y_1, y_2, \ldots, y_n\}$ denotes the maximum out of these numbers

(D) $E^* = E \cup \{e^*\}$ is a basic set E supplemented by the special element e^*

(E) U is defined as a sum of y_j characterising these elements e_j that precede the special element e^* for a given ordering of elements of the set E^*

(F) The probability function assigns to each ordering of the set E^* the probability $1/(n+1)!$

Under assumptions (A)-(F) the random variable U has on the support $[0,1]$ a discrete distribution that is uniquely determined by the set of real numbers $\{y_1, y_2, ..., y_n\}$. More formally, the distribution is determined by the set E and the function y. Let us denote the cdf of this distribution by F_U. Now the theorem can be formulated as follows:

Theorem.
Under assumptions (A)-(F) the cdf F_U is uniformly bounded from both sides:

$$\forall u \in [0,1] \qquad \frac{u}{1+M} \le F_U(u) \le \frac{u+M}{1+M}.$$

Comment 1.
In lights of the theorem convergence of M to zero means that the distribution converges to the uniform distribution. Convergence of M to zero implies that n tends to infinity, as obviously $Mn \ge 1$. However, for given M the number n is unbounded from above.

Comment 2.
It is obvious that each particular ordering of the set E^* corresponds to the inverted ordering. Hence both variables U and $(1-U)$ must have the same distribution. That is why the upper bound for $F_U(u)$ in the theorem equals the lower bound for $1-F_U(1-u)$ and vice-versa. So it suffices to prove that one of the bounds is true. Below, the proof for the upper bound is presented.

Proof.
The proof is based on induction, and shows that the upper bound assumed to be correct for a random variable U determined by any admissible sequence $\{y_j\}$ containing $(n-1)$ elements implies that it is also correct for a variable U determined by any admissible sequence $\{y_j\}$ that contains n elements. So the first step is to show that the upper bound is correct for $n=1$.

1st step of the proof.
For $n=1$ the (only one) value y_1 equals one, and so $M=1$. The variable U can take only values 0 or 1, each of them with probability ½. So the cdf $F_U(u)$ takes a value ½ on the interval $u \in [0,1)$ and the value 1 at the right endpoint $u=1$. Thus the cdf obviously satisfies the inequality $F_U(u) \le (u+1)/2$ for all $u \in [0,1]$.

2nd step of the proof
For the purpose of this step of the proof let us denote by U the variable determined by the set of elements $\{e_1, e_2, ..., e_n\}$ and the corresponding set of values $\{y_1, y_2, ..., y_n\}$, and

assume that $n \geq 2$. Let us also assume for simplicity that there are no larger values of y_j than the first one, so $y_1 = M$, and let us denote the maximum out of the subsequence $\{y_2,...,y_n\}$ by M_1. Let us also make an additional working assumption:

(G) that at least two elements have non-zero values of y_j, so that $0 < M_1 \leq M < 1$.

Let us also define the corresponding family of n random variables U_j, for each $j = 1,2,...,n$ being determined by the set $E_j \doteq E \setminus \{e_j\}$ and the same function *y*. Of course, assumptions (A)-(F) are satisfied for variables $U_j/(1-y_j)$, and not for the variables U_j themselves. So, assuming that the upper bound stated in the theorem is correct for all variables $U_j/(1-y_j)$ means that for all $u \in [0,1]$ the following inequalities hold:

(7) $$\Pr(U_1 \leq u) \leq \frac{u + M_1}{1 + M_1 - M},$$

(8) $$\Pr(U_j \leq u) \leq \frac{u + M}{1 + M - y_j}, \qquad j = 2,3,...,n.$$

The above upper bounds could be easily tightened (making use for instance of simple remark that a probability never extends one). However, for our purposes it suffices to refine the upper bound for the variable U_1 only. The refinement takes a form:

(9) $$\Pr(U_1 \leq u) \leq u + M.$$

For all $u > 1 - M$ the above inequality is correct as a probability is never greater than one. In the case when $u \leq 1 - M$ its correctness could be deduced as follows:

- $u \leq 1 - M$ implies that: $-1 \leq -(M + u)$;
- but by definition $M \geq M_1$, so multiplying both sides of the last inequality by $(M - M_1)$ we obtain: $(M_1 - M) \leq (M_1 - M)(M + u)$,
- adding on both sides the term $M + u$ we obtain: $M_1 + u \leq (1 + M_1 - M)(M + u)$,
- finally, dividing both sides by $(1 + M_1 - M)$ and combining the result with (7) we obtain the confirmation that the inequality (9) holds.

Now we can return to the distribution of the variable *U*. Let us define in the space of all *n*! orderings of elements of the set *E* the event A_j, that the element e_j has taken the last position in the ordering. The events $A_1, A_2,..., A_n$ defined this way are separate, equally probable, and altogether they cover the whole space. Hence we can write:

(10) $$\Pr(U \leq u) = \frac{1}{n}\sum_{j=1}^{n} \Pr(U \leq u | A_j)$$

However, conditional probabilities appearing on the RHS concerning events expressed in terms of values of the variable U could be expressed as well in terms of values of variables U_j. In order to do that, let us notice that for any given ordering of elements of the set E there are $(n+1)$ corresponding orderings of elements of the extended set E^* that differ only by the position of the special element e^*. When an arbitrary ordering of the set E encompassed by the event A_j takes place, then almost always (with one exception) the variable U is identical to the variable U_j. This exceptional case happens when the special element e^* takes the last position, but then obviously $U = 1$. That is why we can write:

$$(11) \quad \forall u \in [0,1) \qquad \Pr\left(U \le u \middle| A_j\right) = \frac{n}{n+1} \Pr\left(U_j \le u\right), \qquad j = 1,2,\ldots,n.$$

Combining now (10) and (11) we obtain:

$$(12) \quad \forall u \in [0,1) \qquad \Pr(U \le u) = \frac{1}{n+1} \sum_{j=1}^{n} \Pr\left(U_j \le u\right).$$

Making use now of upper bounds (8) and (9) we conclude that:

$$(13) \quad \forall u \in [0,1) \qquad \Pr(U \le u) \le \frac{1}{n+1} \left\{ M + u + \sum_{j=2}^{n} \frac{M+u}{M+1-y_j} \right\}.$$

Taking now into account that for any j the following equality holds:

$$\frac{M+u}{M+1-y_j} = \frac{M+u}{M+1} \left\{ 1 + \frac{y_j}{M+1-y_j} \right\}$$

we can transform (13) to the form:

$$(14) \quad \forall u \in [0,1) \qquad \Pr(U \le u) \le \frac{M+u}{n+1} \left\{ 1 + \frac{1}{M+1} \left[n - 1 + \sum_{j=2}^{n} \frac{y_j}{M+1-y_j} \right] \right\}.$$

Now we can make two remarks:

- that each component of the sum appearing on RHS of inequality (14) can be bounded from above: $y_j / \left(M + 1 - y_j\right) \le y_j$, and:
- that the sum $\left(y_2 + y_3 + \ldots + y_n\right)$ equals $(1 - M)$.

These remarks allow for simplification of the upper bound in (14) to the form:

$$(15) \quad \forall u \in [0,1) \qquad \Pr(U \le u) \le \frac{M+u}{n+1} \left\{ 1 + \frac{1}{M+1} \left[n - 1 + (1 - M) \right] \right\}.$$

Simple manipulations allow now for transforming the RHS to the desired form of the upper bound $(M+u)/(M+1)$, and obviously to extend this result to the right endpoint of the interval $[0,1]$.

Now the only thing that is needed to complete the proof of the theorem is to consider the case when the working assumption (G) is not satisfied. The assumption was necessary to avoid dividing by zero (as in (7), and in the final step of the derivation of (9) for instance). But when the set $\{y_1, y_2, ..., y_n\}$ contains only one positive element so that $y_1 = M = 1$, then the distribution of the variable U is obviously the same as in case of $n = 1$, as in all orderings it is only the mutual positions of e_1 and e^* that do matter. The correctness of the theorem for the case $n = 1$ was already considered in the 1st step of the proof.□

5. CASES WHEN REFINEMENTS OF THE THEOREM ARE NEEDED

Unless we impose additional restrictions on the sequence $\{y_1, y_2, ..., y_n\}$, the bounds stated in the theorem cannot be significantly tightened. This can be shown by considering the "worse" example, when all numbers y_j from the sequence equal $1/n$. Then the cdf takes on the interval $[0,1)$ the form:

$$\forall u \in [(k-1)/n, k/n) \quad F_U(u) = k/(n+1) , \quad k = 1,2,...,n$$

On each subinterval the upper bound is attained at the left endpoint and the lower bound is "almost" attained near the right endpoint. So we cannot tighten bounds except by replacing the inequality symbol ≤ concerning the lower bound in the theorem by the strict inequality symbol <.

Of course, any additional restrictions imposed on the sequence $\{y_1, y_2, ..., y_n\}$ may lead to various improvements of the bounds stated by the theorem. However, the variety of possible cases is unlimited, so there is a need for concentration on these cases, where the bounds stated by the theorem are unsatisfactory.

5.1. IDENTIFICATION OF TROUBLESOME CASES

In order to identify cases that are worth considering, let us inspect first how the theorem works when we wish to bound the premium loading. Let us denote the bounds for the cdf $F_U(u)$ stated by the theorem by:

$$\underline{F}(u) := \frac{u}{1+M} , \text{ and } \overline{F}(u) := \frac{u+M}{1+M} .$$

As cdf's $\underline{F}$, F_U and $\overline{F}$ are stochastically ordered and the function:

$$h(u) := \sqrt{u\sigma_W^2 + \sigma_X^2} - \sqrt{u\sigma_W^2}$$

is decreasing on the interval $[0,1]$, we can express the bounds for the premium loading for the risk X that is added to the basic portfolio W as follows:

$$\int_{[0,1]} h(u) d\underline{F}(u) \leq \int_{[0,1]} h(u) dF_U(u) \leq \int_{[0,1]} h(u) d\overline{F}(u) .$$

In order to justify approximation of the loading by the expression σ_X^2 / σ_W, we should inspect whether the bounds divided by this expression are close to unity. Denoting (as before) the ratio σ_X^2 / σ_W^2 by c and executing all required calculations we obtain for the lower bound:

$$\frac{\sigma_W}{\sigma_X^2}\int_{[0,1]} h(u)d\underline{F}(u)=\frac{M\left(\sqrt{1+c}-1\right)+\frac{2}{3}\left[(1+c)^{3/2}-1-c^{3/2}\right]}{c(1+M)}$$

and for the upper bound:

$$\frac{\sigma_W}{\sigma_X^2}\int_{[0,1]} h(u)d\overline{F}(u)=\frac{M\sqrt{c}+\frac{2}{3}\left[(1+c)^{3/2}-1-c^{3/2}\right]}{c(1+M)}$$

Now we can consider various scenarios.

- The scenario when we price a substantially large risk on the background of the portfolio of very numerous small risks can be inspected by assuming fixed c and M tending to zero. Then both bounds tend to the same function $c^{-1}\left[(1+c)^{3/2}-1-c^{3/2}\right]$ that is close to one for c reasonably small.
- There is still no problem when we assume that the size of the priced risk X is comparable to the size of the largest risk, so that $c=const\cdot M$, and again assume that M tends to zero. Also in this case both bounds tend to one.
- The problem arises when we allow for some large risks in the portfolio and try to price risks that are incomparably smaller. This can be reflected by the assumption that M is fixed and c tends to zero. In this case the lower bound tends to $\left(1+\frac{1}{2}M\right)(1+M)^{-1}$, that is still acceptable (at least for small M), but the upper bound tends to infinity, that is no longer acceptable.

The above considerations lead to the conclusion that the problem is only partially solved. Lower bounds work almost satisfactorily in any case, and could be used at least to allocate the dominant part of the whole-portfolio loading. However, diverging upper bounds for risks incomparably smaller than the largest one give no indication how to allocate the lasting part of the loading, and how much the final result can differ from the postulated allocation proportional to the variance.

5.2. MIXED GAMES WITH FEW ATOMS AND THE OCEAN

The case that has been identified as troublesome corresponds to so called mixed games, where some relatively large players are explicitly treated as "atoms" whereas others compose the non-atomic part of the set of players, sometimes called "ocean". In order to have a closer look onto this case, we introduce some additional notations:

- S - the share of n atoms (limited number of large risks) in the variance of the whole portfolio, so that $S=y_1+\ldots+y_n$, and $(1-S)$ corresponds to the share of ocean (very large number of very small risks)
- $\beta \doteq \begin{bmatrix}\beta_1 & \beta_2 & \ldots & \beta_n\end{bmatrix}'$ - the column vector of random variables such that β_j equals one when the atom number j precedes the element e^* in the randomly drawn ordering, and zero otherwise
- $y \doteq \begin{bmatrix}y_1 & y_2 & \ldots & y_n\end{bmatrix}'$ - the column vector of y_j
- l - the n-element column vector of ones
- $B \doteq \{0,1\}^n$ - the support for the random vector β

Now the process of random location of the additional element e^* in the given order of players can be reconstructed as the two-stage experiment:

- at the first stage the element e^* is randomly located among the risks composing the ocean, with the resulting relative weight of preceding elements equal V (random variable uniformly distributed over the unit interval)
- at the second stage "atoms" are located in the same manner as the element e^*, so that each of them precedes e^* with probability V

Given the result of the first stage, the conditional distribution of the variable U given V can be expressed as follows:

$$(16)\quad \Pr\left(U = y'\beta + (1-S)v \middle| V = v\right) = v^{l'\beta}(1-v)^{n-l'\beta}, \quad \beta \in B$$

and so the expected value of any function (let us say, g) of the variable U can be expressed as the sum of integrals:

$$(17)\quad \mathrm{E}\left(g(U)\right) = \sum_{\beta \in B} \int_0^1 g\left(y'\beta + (1-S)v\right) v^{l'\beta}(1-v)^{n-l'\beta}\, dv .$$

The loading formula that we are looking for is given by:

$$\sigma_W \mathrm{E}\left(\sqrt{U+c} - \sqrt{U}\right),$$

so that for sufficiently small c it can be represented by:

$$\sigma_W c \mathrm{E}\left(\frac{\sqrt{U+c}-\sqrt{U}}{c}\right) \approx \sigma_W c \mathrm{E}\left(\frac{\partial \sqrt{U}}{\partial U}\right) = \sigma_W c \mathrm{E}\left(\frac{1}{2\sqrt{U}}\right).$$

As the postulated formula for the loading proportional to the variance reads $\sigma_W c$, our question in fact is:

- whether (and when) the expectation of the function $g(U) = \left(2\sqrt{U}\right)^{-1}$ is close enough to one.

Applying formula (17) in the general case can be extremely laborious. In the next subsections only some selected cases are considered: namely, the case of one atom and the case of n atoms of the same size.

5.3. ONE ATOM AND THE OCEAN

In the case of one atom we have $S = M$, and the formula (17) reads:

$$(18)\quad \mathrm{E}\left(\frac{1}{2\sqrt{U}}\right) = \int_0^1 \frac{v\,dv}{2\sqrt{M+(1-M)v}} + \int_0^1 \frac{(1-v)\,dv}{2\sqrt{(1-M)v}}$$

Simple calculations lead to the result:

$$\mathrm{E}\left(\frac{1}{2\sqrt{U}}\right) = \frac{\frac{1}{3} - M + \frac{2}{3}(1-M)^{3/2} + \frac{2}{3}M^{3/2}}{(1-M)^2}$$

The result for $M = 1/10$ is (approximately) 1.017, and even for M as large as 25% is still moderate and equals 1.066.

By the way, we can obtain explicit formula for the density of the variable U by substituting $(1-M)v + M$ by u in the first integral and $(1-M)v$ by u in the second integral. The resulting formula for $u \in (0,1)$ reads:

$$f_U(u) = \frac{(1-M-u)_+ + (u-M)_+}{(1-M)^2}.$$

It is worth noting that in this case (as in general in case of mixed games), there is no more probability mass at the endpoints of the unit interval. This is essentially the main advantage of the approach based on mixed games as compared to bounding the cdf F_U by $\underline{F}$ and $\overline{F}$, according to the theorem.

5.4. n ATOMS OF THE SAME SIZE AND THE OCEAN

In this case formula (17) applied to the function $g(U) = \left(2\sqrt{U}\right)^{-1}$ simplifies to the form:

$$\text{(19)}\quad \mathrm{E}\left(\frac{1}{2\sqrt{U}}\right) = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 \frac{v^k (1-v)^{n-k}}{2\sqrt{S\frac{k}{n} + (1-S)v}}\, dv.$$

Direct calculations based on the above formula can be extremely laborious for n substantially greater than one. However, upper bounds for the expectation could be derived. In order to obtain them it is convenient to transform the formula to the form:

$$\text{(20)}\quad \mathrm{E}\left(\frac{1}{2\sqrt{U}}\right) = \frac{1}{2(n+1)} \sum_{k=0}^{n} \int_0^1 \frac{f_k(v)}{\sqrt{S\frac{k}{n} + (1-S)v}}\, dv,$$

where f_k is the density of the beta distribution with parameters $(k+1, n-k+1)$. Thus all integrals appearing on the RHS can be interpreted as expected values of functions $g_k(V_k) = \left(S\frac{k}{n} + (1-S)V_k\right)^{-1/2}$ of random variables V_k having Beta distributions with parameters $(k+1, n-k+1)$.

Direct calculations are easy only for the first term of the sum:

$$\text{(21)}\quad \mathrm{E}\{g_0(V_0)\} = \frac{1}{\sqrt{(1-S)}} \int_0^1 \frac{f_0(v)}{\sqrt{v}}\, dv =$$

$$= \frac{1}{\sqrt{(1-S)}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \int_0^1 \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)} (1-v)^n v^{-1/2} dv =$$

$$= \frac{\sqrt{\pi}}{\sqrt{(1-S)}} \frac{\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}.$$

For large n the following approximation can be used:

$$\text{(22)}\quad \mathrm{E}\{g_0(V_0)\} \approx \frac{\sqrt{\pi}}{\sqrt{(1-S)}} \cdot \sqrt{n+\tfrac{5}{4}}$$

Calculations for other terms of the sum are cumbersome. However, all of them can be bounded from above. The bounds can be based on the refined version of the Taylor expansion of the function $f(x) = x^{-1/2}$ around the point $x = 1$. Neglecting terms of order greater than two of the expansion we obtain:

$$x^{-1/2} \approx 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 .$$

The upper bound related to the above approximation is based on enlargement of the coefficient 3/8 that multiplies the quadratic term of the expansion. First of all it can be easily shown that for arbitrary positive real number x the following inequality is true:

$$\text{(23)} \qquad \frac{1}{\sqrt{x}} \le 1 - \frac{1}{2}(x-1) + \frac{1}{2x}(x-1)^2 .$$

In order to do that we can re-express the RHS:

$$RHS = 1 - \frac{1}{2}(x-1) + \frac{1}{2x}(x-1)^2 = 1 + \frac{1}{2}(x-1)\left[-1 + \frac{x-1}{x}\right] =$$

$$= 1 + \frac{1}{2}(x-1)\left[-1 + \frac{x-1}{x}\right] = \frac{1}{2}\left(1 + \frac{1}{x}\right).$$

But the inequality $x^{-1/2} \le \frac{1}{2}\left(1 + x^{-1}\right)$ is obviously true as $\left(1 - x^{-1/2}\right)^2 \ge 0$. □

Let us assume now, that all considered values of the variable x are no smaller than some positive number x_0. Under this assumption we can replace the variable coefficient $(2x)^{-1}$ appearing on the RHS of (23) by the constant coefficient $(2x_0)^{-1}$ in order to obtain finally the upper bound:

$$\text{(24)} \quad \frac{1}{\sqrt{x}} \le 1 - \frac{1}{2}(x-1) + \frac{1}{2x_0}(x-1)^2$$

Now we can replace numbers x and x_0 appearing in the inequality (24) by:

$$x := \frac{n}{k}\left(S\tfrac{k}{n} + (1-S)v\right) \quad \text{and} \quad x_0 := S .$$

This replacement satisfies the condition $x \ge x_0 > 0$ for arbitrary non-negative number v and $S \in (0,1)$. After the replacement and some rearrangements that makes the result more readable the inequality (24) takes the following form:

$$\frac{1}{\sqrt{S\tfrac{k}{n} + (1-S)v}} \le \sqrt{\frac{n}{k}}\left\{1 - \frac{(1-S)}{2}\frac{n}{k}\left(v - \frac{k}{n}\right) + \frac{(1-S)^2}{2S}\frac{n^2}{k^2}\left(v - \frac{k}{n}\right)^2\right\}.$$

Replacing now the number v by the random variable V_k and taking expectations on both sides of the above inequality we obtain upper bounds for integrals that for $k = 1,2,...,n$ appear on the RHS of (20):

$$\mathrm{E}\{g_k(V_k)\} \le \sqrt{\frac{n}{k}}\left\{1 - \frac{1-S}{2}\frac{n}{k}\mathrm{E}\left(V_k - \frac{k}{n}\right) + \frac{(1-S)^2}{2S}\frac{n^2}{k^2}\mathrm{E}\left(V_k - \frac{k}{n}\right)^2\right\}.$$

As $\mathrm{E}(V_k)=\dfrac{k+1}{n+2}$, and $\mathrm{Var}(V_k)=\dfrac{(k+1)(n-k+1)}{(n+2)^2(n+3)}$, we obtain:

$$\mathrm{E}\{g_k(V_k)\}\le\sqrt{\frac{n}{k}}\left\{1-\frac{1-S}{2}\frac{n-2k}{(n+2)k}+\frac{(1-S)^2}{2S}\frac{2n^2+(n-6)(n-k)k}{k^2(n+2)(n+3)}\right\}.$$

For given *n* and *S* the RHS of the last inequality is in fact a linear combination of terms $k^{-1/2}$, $k^{-3/2}$, and $k^{-5/2}$. Summing both sides over the range $k=1,2,...,n$ we obtain:

$$\text{(25)}\quad \sum_{k=1}^{n}\mathrm{E}\{g_k(V_k)\}\le a_1\sum_{k=1}^{n}\frac{1}{\sqrt{k}}+a_2\sum_{k=1}^{n}\frac{1}{k\sqrt{k}}+a_3\sum_{k=1}^{n}\frac{1}{k^2\sqrt{k}},\quad \text{where:}$$

$$a_1=\frac{\sqrt{n}}{(n+2)(n+3)}\left\{(n+3-S)(n+3)-\frac{(1-S)^2}{2S}(n-6)\right\},$$

$$a_2=\frac{\sqrt{n}}{(n+2)(n+3)}\left\{-\frac{1-S}{2}n(n+3)-\frac{(1-S)^2}{2S}6n+\frac{(1-S)^2}{2S}n^2\right\},$$

$$a_3=\frac{\sqrt{n}}{(n+2)(n+3)}\frac{(1-S)^2}{S}n^2.$$

Formulas (21) and (25) can be used for direct numerical calculations of upper bounds for given *S* and *n*. In order to obtain some approximations for large *n* further simplifications are needed.

There are no simple formulas for the sums $\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$, $\sum_{k=1}^{n}\frac{1}{k\sqrt{k}}$, and $\sum_{k=1}^{n}\frac{1}{k^2\sqrt{k}}$. However, it is quite easy to find quite tight bounds for them. The term a_3 is always positive, thus it is sufficient to find the upper bound for the third sum. The coefficient a_1 can be negative for some *n*, but only when $S<1.4\%$. Thus also for this case we will rely on the upper bound for the first sum. The term a_2 can be as well positive as negative in relevant cases, so we will multiply negative components of this term by the lower bound for the second sum, and positive term by the upper bound. The upper bounds are as follows:

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\le\int_{1/2}^{n+1/2}x^{-1/2}dx=2\left(\sqrt{n+\tfrac{1}{2}}-\sqrt{\tfrac{1}{2}}\right)$$

$$\sum_{k=1}^{n}\frac{1}{k\sqrt{k}}\le\int_{1/2}^{n+1/2}x^{-3/2}dx=2\left(\sqrt{2}-\left(n+\tfrac{1}{2}\right)^{-1/2}\right)$$

$$\sum_{k=1}^{n}\frac{1}{k^2\sqrt{k}}\le\int_{1/2}^{n+1/2}x^{-5/2}dx=\frac{2}{3}\left(2\sqrt{2}-\left(n+\tfrac{1}{2}\right)^{-3/2}\right)$$

All of them can be justified by convexity of integrated functions. Much less sophisticated is the lower bound for the second sum:

$$\sum_{k=1}^{n}\frac{1}{k\sqrt{k}}\ge\int_{1}^{n+1}x^{-3/2}dx=2\left(1-(n+1)^{-1/2}\right).$$

Collecting all results for $\mathrm{E}\{g_k(V_k)\}$ over $k=0,1,2,...,n$, and taking only terms of the highest two orders we obtain an approximated (because of neglecting terms of order $n^{-1},n^{-3/2},n^{-2},...$) inequality:

$$(26)\quad \mathrm{E}\left(\frac{1}{2\sqrt{U}}\right) \approx\leq \frac{\sqrt{n\left(n+\frac{1}{2}\right)}}{n+1} + \\ + \frac{1}{2(n+1)}\left\{\frac{\sqrt{\pi\left(n+\frac{5}{4}\right)}}{\sqrt{1-S}} - \sqrt{2n} + \frac{n\sqrt{n}}{n+2}\left[\frac{7\sqrt{2}(1-S)^2 n}{3S(n+3)} - (1-S)\right]\right\}.$$

Of course, the RHS of the last formula tends to one for $n \to \infty$. However, the convergence need not to be very fast because of substantial terms proportional to $n^{-1/2}$.

The results for the case of $S = 1/2$ are presented in the table below. Exact calculations are shown in column 2, only for the cases of $n = 1$ (formula 18) and $n = 2$ (detailed derivation based on formula (17) is not presented in the paper). Upper bounds based on formulas (21) and (25) are presented for $n = 1,2,...,10$ in the column 3. Approximations based on formula (26) are presented for $n = 1,2,...,10$ and selected larger values of *n* in column 4.

Table 1. Expected value of $\left(2\sqrt{U}\right)^{-1}$: exact, upper bounds and their approximations

n	exact	up.bound	approx.
1	121.9%	122.4%	119.4%
2	117.4%	118.3%	118.5%
3	-	116.8%	119.3%
4	-	116.1%	119.9%
5	-	115.8%	120.4%
6	-	115.6%	120.6%
7	-	115.4%	120.6%
8	-	115.2%	120.6%
9	-	115.1%	120.4%
10	-	114.9%	120.3%
25	-	-	116.9%
50	-	-	113.4%
100	-	-	110.1%
250	-	-	106.7%
500	-	-	104.8%
1000	-	-	103.5%
10000	-	-	101.1%

In the two cases when exact values are available, upper bounds seem to be quite tight. On the other hand, approximations seem to be rather rough. Nonetheless they confirm that the expectation of $\left(2\sqrt{U}\right)^{-1}$ tends to one as *n* tends to infinity.

6. CONCLUSIONS

The results presented in sections 4 and 5 mean that basically the variance principle can be justified as an approximation to the Shapley value. In the worse case when the portfolio contains both large risks and the ocean of very small risks, the Shapley loading for small risks is moderately greater than that proportional to the variance. The content of the subsection 5.4 illustrates rather the worse case because of two reasons:

- One is that the upper bounds overestimate the true numbers.
- The second is that not all large risks are usually as large as the largest, and the largest exhaust usually no more than few percentage points of the total variance. Otherwise it is highly doubtful that the aggregate amount of claims is approximately normal, and the premium formula applied for the portfolio should be based on other principles than the standard deviation principle.

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