

Dynamic life tables. Age-period-cohort models

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Abstract

Dynamic tables arise from the necessity to incorporate the effect of the year of death (period) into the estimation of the measurement of mortality. They have marked an important advance with respect to statistical tables in as much as they allow us to capture the historical evolution of mortality. For this reason they have become a very appropriate tool for actuarial science.

In their book, Tabeau et al. (2001) describe the recently developed models and conclude with the need to integrate the techniques of distinct disciplines with the aim of achieving satisfactory predictions. In Mortality sub-committee (2004) it is recommended that actuaries adopt age-period-cohort models which have given good results in the field of epidemiology. These models constitute a natural evolution of dynamic models as they incorporate the year of birth (cohort). The objective of our paper is to use the models suggested by Holford (1983) and Clayton and Schifflers (1987a,b) to adjust mortality in Spain, Sweden and the Czech Republic and study the effect of the cohort in the three countries.

Keywords: Standard life tables, Dynamic life tables, APC models.

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1 Introduction

The prediction of mortality is used in many countries to create and modify retirement pensions, disability insurance and social security programmes. The spectacular growth in life expectancy throughout the 20th century, due without doubt to advances in the world of medicine and of experimental sciences, has made it necessary to improve the methods of adjustment and prediction of mortality due to its social and financial repercussions.

Tabeau et al. (2001) conclude “... that satisfactory forecasting requires the integration of perspectives and techniques from the disciplines of statistics, demography and epidemiology”. In line with that, the Mortality sub-committee (2004) recommends that actuaries adopt age-period-cohort models which have given good results in epidemiology. These models incorporate the influence of the year of birth (cohort) into dynamic mortality tables, which in turn arise from the need to incorporate the historical evolution of mortality for each age. We must mention, for its interest, the work by Renshaw and Haberman (2006) which extends the models based on factors of mortality reduction (Renshaw and Haberman, 2003b,a), incorporating the effect of the cohort, using the structure by Lee and Lee and Carter (1992) to do that.

The content of our paper is structured as follows. The description of the models used is presented in Section 2. Section 3 is dedicated to the adjustment, through different models, of the crude probabilities of death obtained from the mortality data of three countries in our geographical and social area: Spain, Sweden and the Czech Republic. The last section compares the results obtained for each model and country and extracts the conclusions.

2 Models

The models we are going to use are those described in Carstensen and Keiding (2005). In those notes the authors include functions to use with the R Development Core Team (2005) and adjust the mortality force, μ_{ap} , under a Poisson distribution. We have adapted these functions to adjust the probabilities of death, q_{ap} , under the Binomial distribution.

Mortality and population data are normally registered in matrix form where the rows and columns correspond to age and calendar year respectively. In this way the diagonal corresponds to the cohorts, which are the individuals with the same year of birth.

2.1 Age-period models (AP)

Age-period models establish that the logit of the probability of death depend on age and period through the expression

$$\text{logit}(q_{ap}) = \alpha_a + \beta_p. \quad (1)$$

This model has a parameter for each age group and a parameter for each period and is determined in the absence of a constant which can be added to the α 's, on the condition that the same constant be subtracted from the β 's.

The natural restriction from the epidemiological point of view is to set the parameters of one of the periods equal to zero, $\beta_{p_0}=0$. Then, for the period in question we can write the expression (1) in the form,

$$\text{logit}(q_{ap_0}) = \alpha_a + \beta_{p_0} = \alpha_a, \quad (2)$$

in such a way that the α 's are the logit of the probabilities of death for each age of the period p_0 and, consequently

$$q_{ap_0} = \frac{\exp \alpha_a}{1 + \exp \alpha_a}.$$

Comparing the odds of the probabilities of death in any period p with those of the reference period p_0 , we have

$$\log(OR) = \log \frac{q_{ap}/(1 - q_{ap})}{q_{ap_0}/(1 - q_{ap_0})} = \alpha_a + \beta_p - \alpha_a = \beta_p, \quad (3)$$

for which the β 's are the logs of the ratio odds relative to the probabilities of death in the period p and the period p_0 . These values are the same for all age groups.

The results of this model are two curves: the probabilities of death for each age of the reference period p_0 and the ratio odds of the probabilities of death in the period p and the period p_0 .

2.2 Age-cohort models (AC)

The age-cohort model (AC) is similar to the AP model. Now the logit of the probabilities of death depend on age and the cohort through the expression

$$\text{logit}(q_{ac}) = \alpha_a + \gamma_c. \quad (4)$$

As in the model (1), there is now also a parameter for each age group and a parameter for each cohort and the model is determined in the absence of a constant which can be added to the α 's, on the condition that the same constant be subtracted from the γ 's.

The natural restriction from the epidemiological point of view is fixing the parameter of one of the cohorts equal to zero, $\gamma_{c_0}=0$. Thus, for this cohort we can write the expression (4) in the form

$$\text{logit}(q_{ac_0}) = \alpha_a + \gamma_{c_0} = \alpha_a, \quad (5)$$

such that the α 's are the logit of the probabilities of death for each age of the cohort c_0 and consequently,

$$q_{ac_0} = \frac{\exp \alpha_a}{1 + \exp \alpha_a}.$$

Comparing the odds of the probabilities of death in any cohort with those of the reference cohort c_0 , we have

$$\log(OR) = \log \frac{q_{ac}/(1 - q_{ac})}{q_{ac_0}/(1 - q_{ac_0})} = \alpha_a + \gamma_c - \alpha_a = \gamma_c, \quad (6)$$

for which the γ 's are the logs of the ratio odds relative to the probabilities of death in the cohort c and the cohort c_0 . These values are the same for all age groups.

The results of this model are two curves: the probabilities of death for each age of the reference cohort c_0 and the ratio odds of the probabilities of death of the cohort c and the cohort c_0 .

2.3 The age-drift model

On occasions the evolution of the β 's over time shows a linear tendency, so it makes sense to consider an AP model in which the β_p are replaced by $\beta(p - p_0)$, which supposes a simplification and a reduction of the parameters to estimate. It is a matter of adjusting the model

$$\text{logit}(q_{ap}) = \alpha_a + \beta(p - p_0). \quad (7)$$

The same approximation can also be considered in the AC model, producing $\gamma_c = \gamma(c - c_0)$,

$$\text{logit}(q_{ac}) = \tilde{\alpha}_a + \gamma(c - c_0). \quad (8)$$

Analytically, we can see that the two models are the same, so using the relationship $a = p + c$, we obtain $p_0 = a_0 + c_0$, in such a way that substituting in the expression (7)

$$\alpha_a + \beta(p - p_0) = \alpha_a + \beta(a + c - a_0 - c_0) = \alpha_a + \beta(a - a_0) + \beta(c - c_0),$$

and $\tilde{\alpha}_a = \alpha_a + \beta(a - a_0)$. The interpretation of the parameters in both models are different.

2.4 Age-period-cohort models (APC)

The application of APC models to actuarial problems comes from the suggestion by Tabeau et al. (2001) in as much as they have been shown to behave well in other areas as, for example, epidemiology. On the other hand they are a natural extension of the AP and AC models by considering the simultaneous effect of three factors. The effect of age represents the risks associated with different age groups. The effect of period represents the variation in the probabilities of death due to the year of death. The cohort effect is associated with changes in the these probabilities in the groups of individuals with the same year of birth (cohort).

The AP model has $1 + (A - 1) + P(P - 1)$ parameters, if we add the cohort effect, we will find $C - 2$ parameters and not $C - 1$ as we would expect. In an analogous way, if we begin with the model AC, which has $1 + (A - 1) + (P - 1)$ parameters, and we add the period factor, we will have $P - 2$ new parameters and not $P - 1$. This is due to the relationship which connects the three factors, $a = p - c$. The exact linear dependency between the three factors which this equality supposes is the greatest problem presented by APC models, and is known as the problem of identifiability. There are different solutions to that problem, and the one proposed by Holford (1983) consists of first adjusting the model with any parametrization of the effects, and later carrying out a regression of each set of estimations over its corresponding factor. Let us suppose that we adjust a model with a certain combination of parameters

$$\text{logit}[q(a, p)] = \alpha_a + \beta_p + \gamma_c. \quad (9)$$

The equation is the equivalent if we add to each parameter any value μ_a , μ_p and μ_c in such a way that $\mu_a + \mu_p + \mu_c = 0$. Let us also bear in mind that the APC model verifies the relationship $a = p - c$ and therefore, for any constant δ we will have $\delta(a - p + c) = 0$. Incorporating both equalities to (9), we will obtain

$$\begin{aligned} \text{logit}[q(a, p)] &= \alpha_a - \mu_p - \mu_c + \delta a + \\ &\quad \beta_p + \mu_p - \delta p + \\ &\quad \gamma_c + \mu_c + \delta c \end{aligned} \quad (10)$$

This expression suggests a decomposition of the effects into one linear part and another non-linear, therefore Holford (1983) suggests to carry out a regression of the estimations of each effect over its factor in order to get, once the calculated regressions have been substituted in (9), a new expression similar to (10)

$$\begin{aligned} \text{logit}[q(a, p)] &= \tilde{\alpha}_a + \hat{\mu}_a + \hat{\delta}_a a + \\ &\quad \tilde{\beta}_p + \hat{\mu}_p + \hat{\delta}_p p + \\ &\quad \tilde{\gamma}_c + \hat{\mu}_c + \hat{\delta}_c c. \end{aligned} \quad (11)$$

What we expect is that all the slopes will be the same and equal to the common theoretical slope δ . This problem can be solved formulating a hypothesis on the relative importance of the effects. For example, if we decide that the age scale is the most important and that the period scale is the least, we can choose a reparametrization based on the following hypothesis:

- The period effects should be zero on average.
- The cohort effects should be a relative risk to a central cohort.
- The age effects should represent the probabilities of death for each age in the central reference cohort after correction for the period effects which should be zero on average.

Beginning with the first point, from equation (11) on we can use as period effect

$$g(p) = \tilde{\beta}_p = \beta_p - \hat{\mu}_p - \hat{\delta}_p p,$$

precisely because the residuals are zero on average. On substituting in (11)

$$\begin{aligned} \text{logit}[q(a, p)] &= \tilde{\alpha}_a + \hat{\mu}_a + \hat{\delta}_a a + g(p) + \hat{\mu}_p + \hat{\delta}_p p + \tilde{\gamma}_c + \hat{\mu}_c + \hat{\delta}_c c \\ &= \alpha_a + g(p) + \hat{\mu}_p + \hat{\delta}_p p + \gamma_c, \end{aligned} \quad (12)$$

adding and subtracting the reference cohort, c_0 , and its effect γ_{c_0} and bearing in mind that $p = c + a$, we will have

$$\begin{aligned} \text{logit}[q(a, p)] &= [\alpha_a + \hat{\mu}_p + \hat{\delta}_p(a + c_0) + \gamma_{c_0}] + g(p) + [\gamma_c - \gamma_{c_0} + \hat{\delta}_p(c - c_0)] \\ &= f(a) + g(p) + h(c) \end{aligned} \quad (13)$$

where

$$h(c) = \gamma_c - \gamma_{c_0} + \hat{\delta}_p(c - c_0)$$

is the cohort effect, which is zero for c_0 and has the correct slope, and

$$f(a) = \alpha_a + \hat{\mu}_p + \hat{\delta}_p(a + c_0) + \gamma_{c_0}$$

is the age effect.

Another solution is that proposed by the sequential method: first adjust the AC model and then adjust a model at the residuals only with the period. Distinct from Holford (1983) proposal, this sequential method allows the construction of confidence intervals for the estimated effects. Further details regarding both methods can be found in Carstensen and Keiding (2005).

3 Application of the models

3.1 Data

The models described in Section 2 have been used to adjust the mortality data for Spain, Sweden and the Czech Republic. The adjustment has been done separately for each sex. The data are taken from the H.M.D. (2005) database and correspond to the period 1980-1999 for the age ranges from 0 to 105 years old. The crude death probabilities which the model requires have been obtained through the procedure described in the H.M.D. (2005) protocol.

3.2 Results and conclusions

Following an order of growing complexity, in Tables 1 and 2 it can be seen that for both sexes the models of the two factors, AC and AP, adjust much better in the three countries than the linear tendency (drift) model. Also the model of three factors, APC, in its two methods of adjustment, Holford and sequential, behave better for both sexes and the three countries than the models of two factors. Both tables are divided into three parts to compare the $\Delta Deviance$ of the more simple models with the more complex ones with the objective of checking if the greater number of parameters leads to a significant improvement in the adjustment. The first part compares the A-drift model with the AP and AC models, the second and third parts compare the AP and AC models, respectively, with the two methods of adjustment of APC model. All the Deviance variations are significant.

It is also worth pointing out that the sequential method to adjust the APC model produces very similar results to the Holford method, but has the advantage of providing confidence intervals for the parameter estimations and being simpler to obtain.

To continue with the interpretation of the results it is useful to carry out a graphical statistical analysis of the distinct models.

AP Model Figures 1 and 4 show the behaviour of (2) (left hand column) and the evolution of (3) (right hand column) when the reference period is the year 1989. It can be seen that for both sexes infant mortality improves with the country's development. For very advanced ages mortality decreases in the three countries which, as Pitacco (2004) has already pointed out, occurs in many countries. With respect to its evolution over the 20th century, it can be seen that the downward slope is more or less constant in the case of Spain and Sweden, though more pronounced in the latter. The Czech Republic shows a different pattern, with a much sharper descent after 1990.

AC Model Figures 2 and 5 show the behaviour of (5) (left hand column) and the evolution of (6) (right hand column) when the reference cohort is 1937. It can be seen for both sexes that the hump for twenty years old for those who were born in 1937 is more accentuated in Sweden and the Czech Republic than in Spain. With respect to the evolution of the cohorts, a great variability can be seen in the initial cohorts due to the smaller quantity of data available. For the remainder, what must be highlighted is the hump that appears in Spain for both sexes between the cohorts of 1960-70.

APC Model We have shown in black the maximum-likelihood estimations obtained with Holford method, taking the age scale as the most important and that of period as least important and 1937 as the reference cohort according to expression (13). The grey lines are those obtained by the adjustment for the sequential method with their corresponding confidence intervals, both estimations being confused in the graph because, as mentioned earlier, their results are very similar.

It can be seen that that the effect of the cohort on the mortality of both sexes in Spain shows an increment in the generations after approximately 1960, less pronounced in the case of women. This increment cannot be seen in the other two countries where mortality decreases in a regular pattern.

One future line of our work will go in the direction of introducing the cohort effect into the prediction of mortality and the use of bootstrap techniques to obtain the confidence intervals with the Holford method.

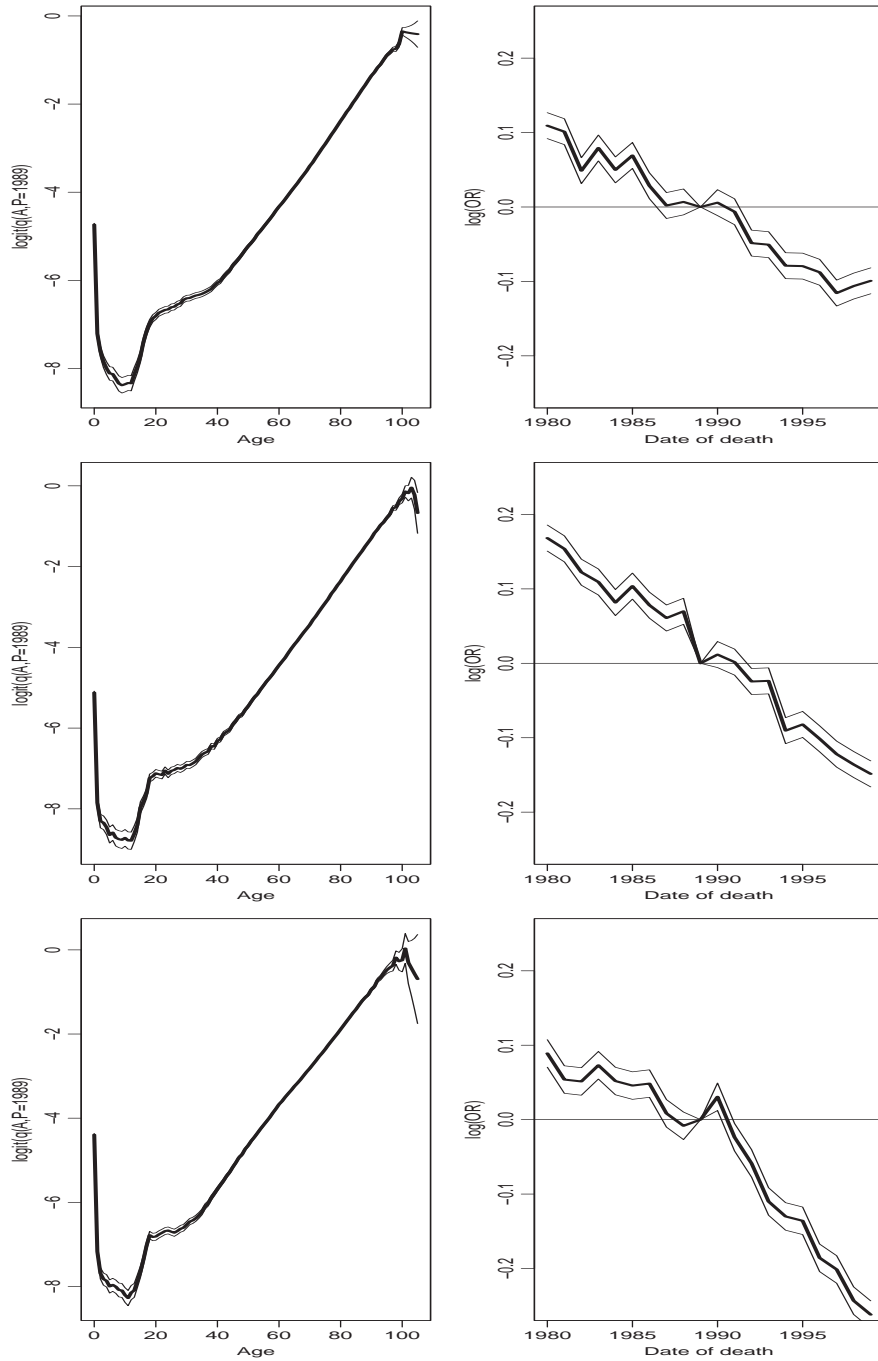


Figure 1: AP model for men for Spain(top), Sweden (middle) and Czech Republic (bottom).

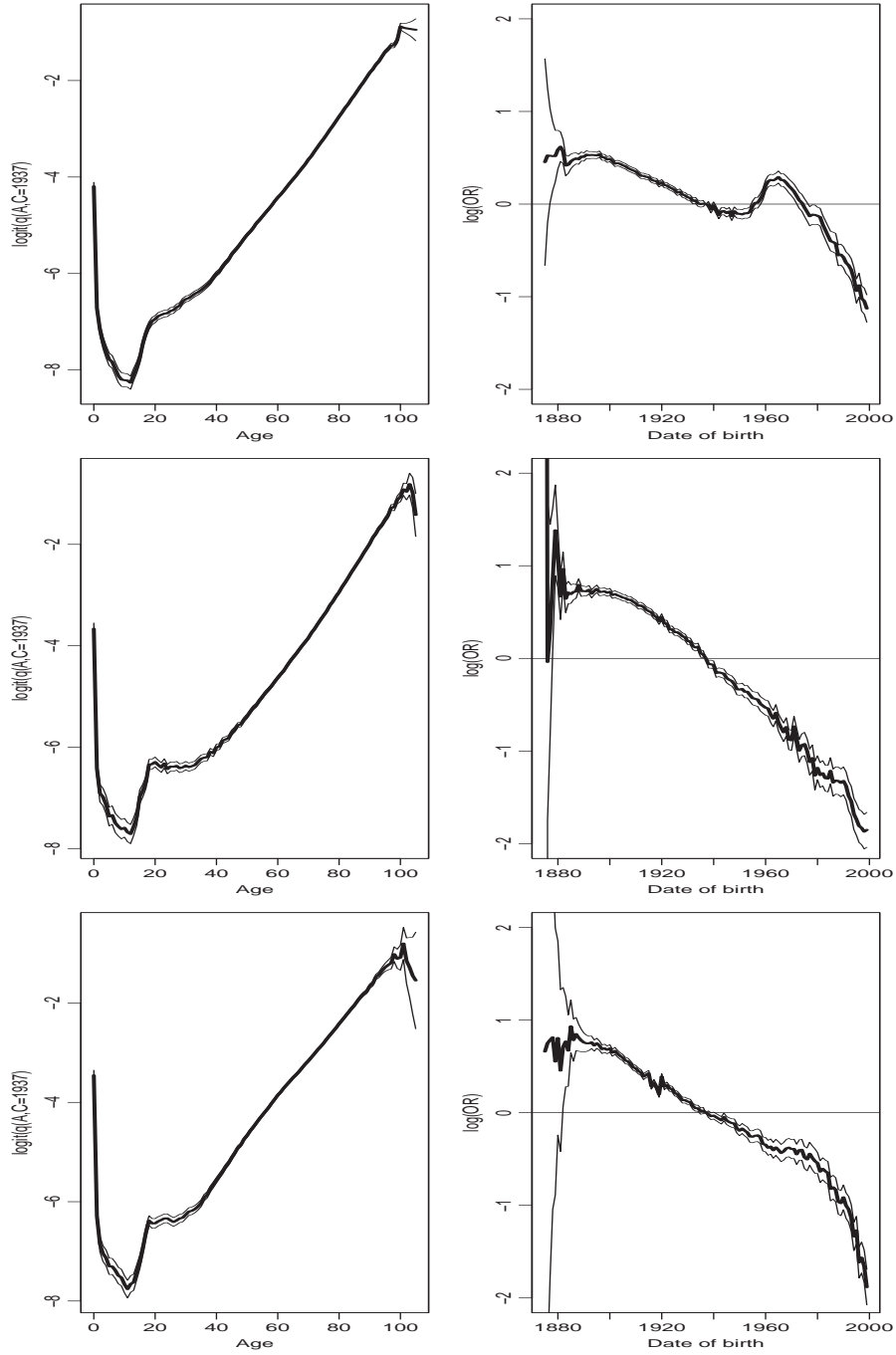


Figure 2: AC model for men for Spain(top), Sweden (middle) and Czech Republic (bottom).

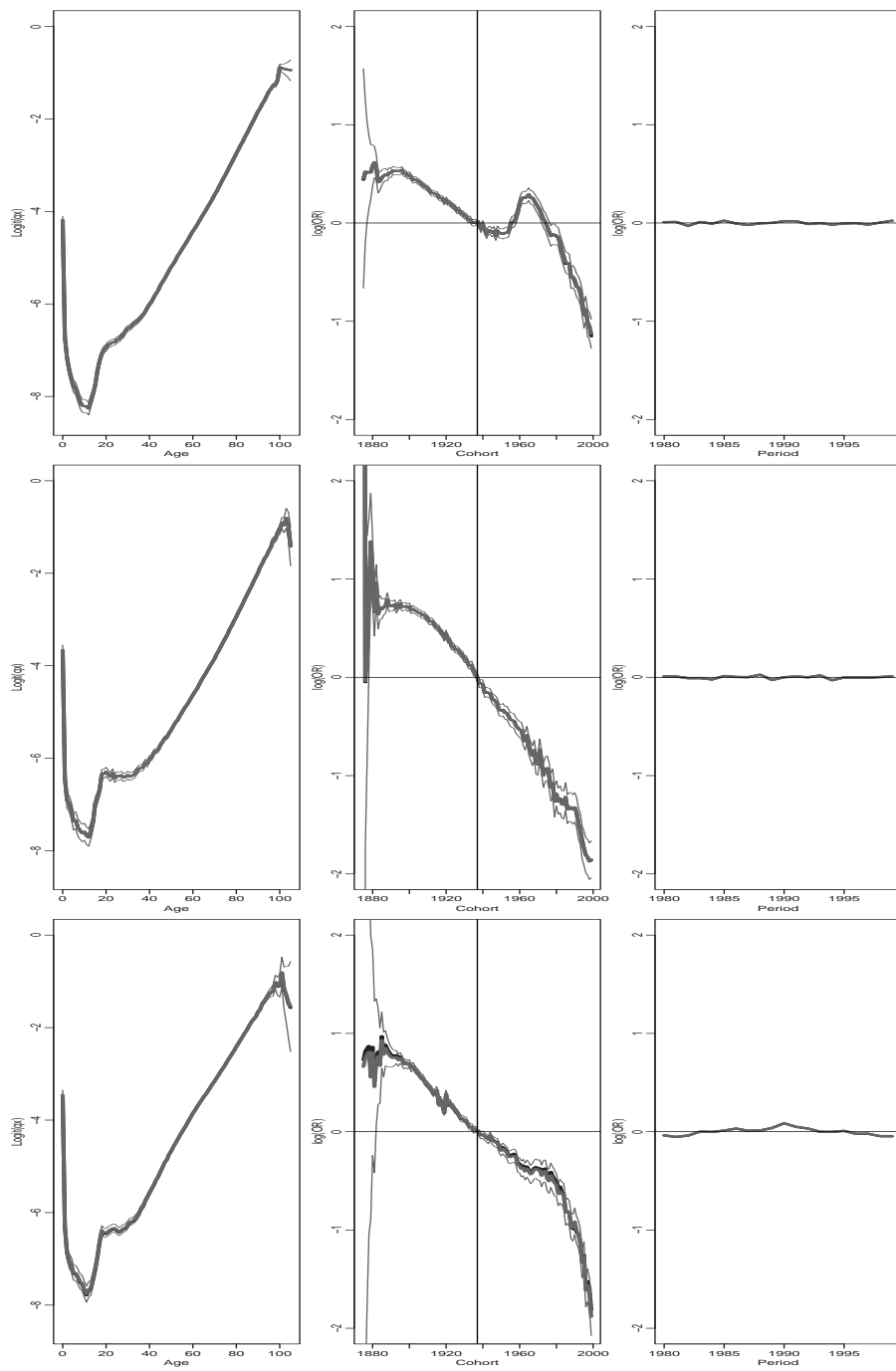


Figure 3: APC model for men for Spain(top), Sweden (middle) and Czech Republic (bottom).

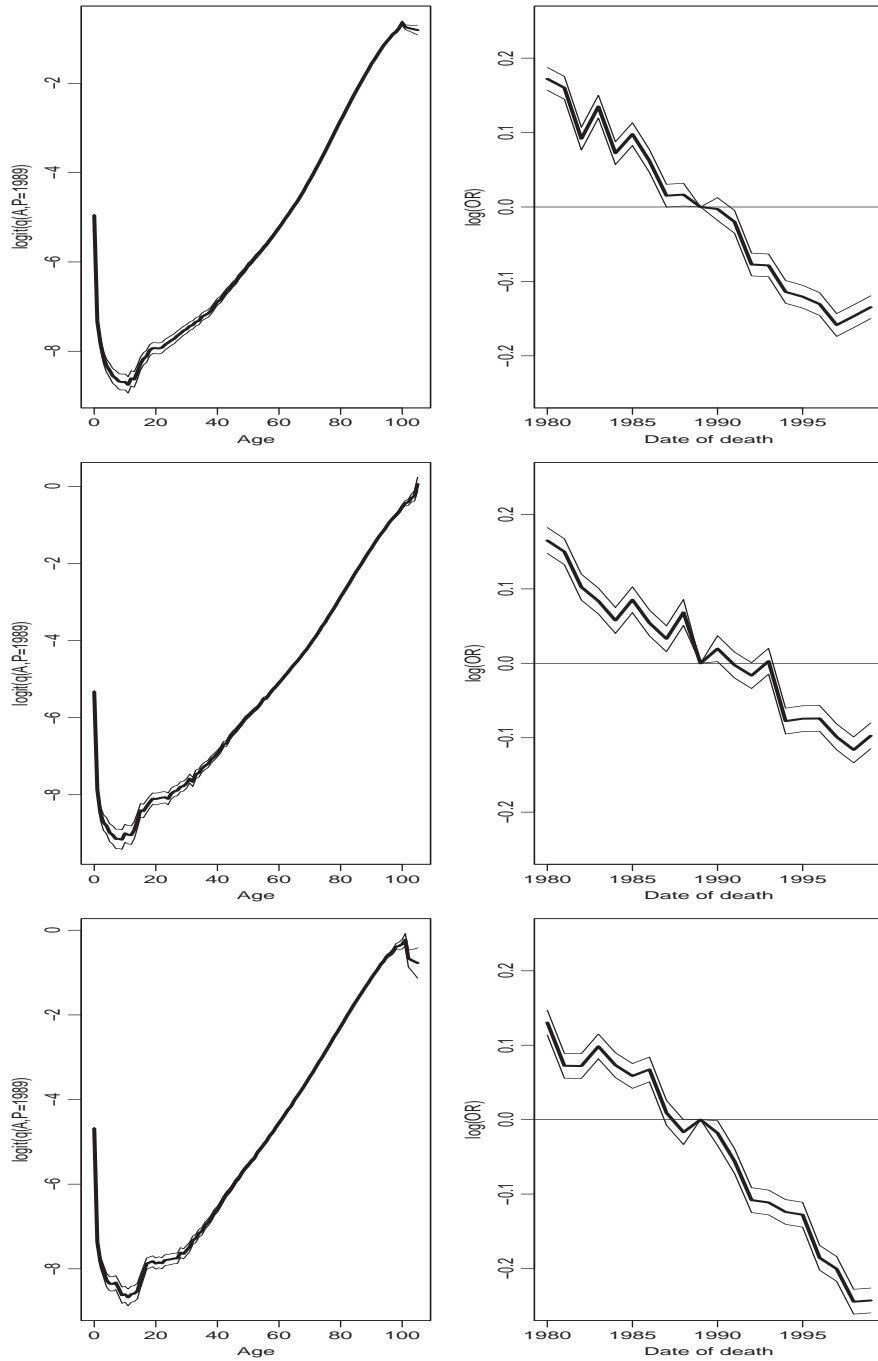


Figure 4: AP model for women for Spain(top), Sweden (middle) and Czech Republic (bottom).

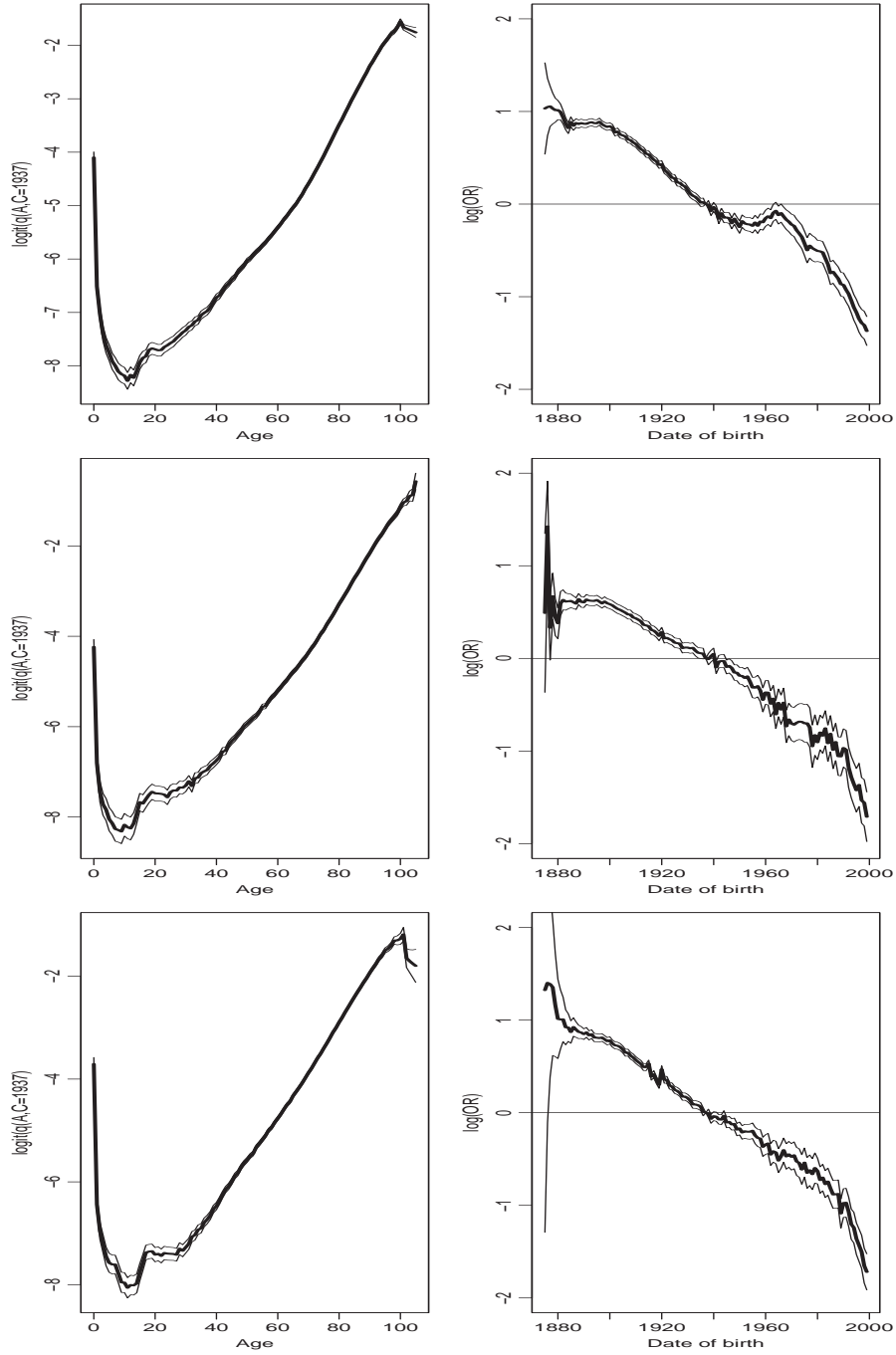


Figure 5: AC model for women for Spain(top), Sweden (middle) and Czech Republic (bottom).

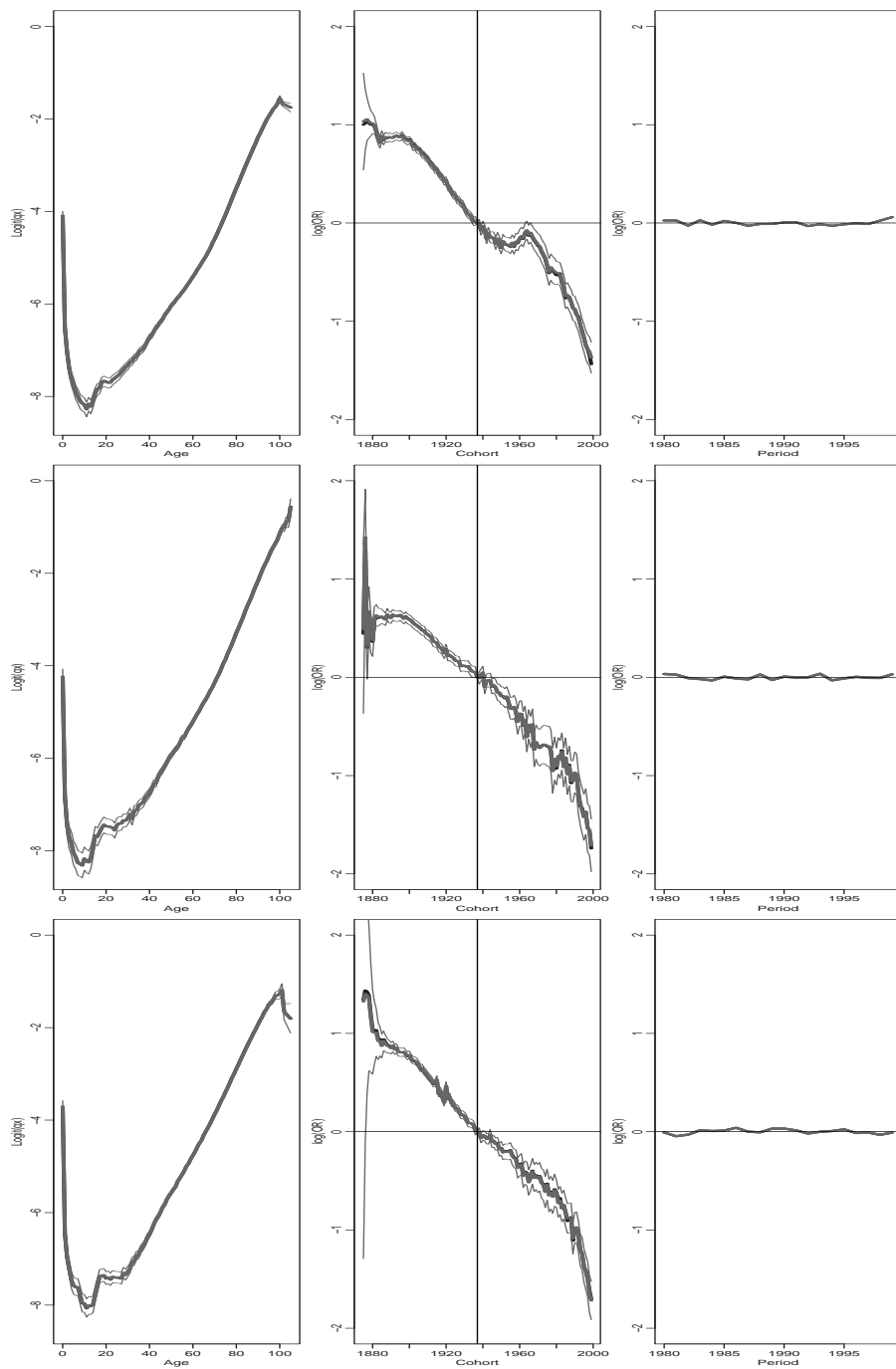


Figure 6: APC model for women for Spain(top), Sweden (middle) and Czech Republic (bottom).

Model	d.f.	Δ d.f.	Spain ^a	Δ Dev.	Sweden ^b	Δ Dev.	Czech R. ^c	Δ Dev.
A-drift	2013		7536.1		7721.8		10474.0	
AP	1995	18	7139.1	397.0	7317.6	404.2	8274.0	2200.0
AC	1890	123	3596.0	3940.1	4544.0	3177.8	6167.5	4306.5
AP	1995		7139.1		7317.6		8274	
APC (Holdford)	1872	123	3195.8	3943.3	4192.4	3125.2	3902.8	4371.2
APC (seq)	1872	123	3198.9	3940.2	4194.2	3123.4	3965.6	4308.4
AC	1890		3596.0		4544.0		6167.5	
APC (Holdford)	1872	18	3195.8	400.2	4192.4	351.6	3902.8	2264.7
APC (seq)	1872	18	3198.9	397.1	4194.2	349.8	3965.6	2201.9

Table 1: Comparison of different models for men

^aover-dispersion parameter $\phi = 1.69$

^bover-dispersion parameter $\phi = 2.21$

^cover-dispersion parameter $\phi = 2.08$

Model	d.f.	Δ d.f.	Spain ^a	Δ Dev.	Sweden ^b	Δ Dev.	Czech R. ^c	Δ Dev.
A-drift	2013		6189.5		7815.0		7582.8	
AP	1995	18	5467.2	722.3	7131.1	683.9	6567.9	1014.9
AC	1890	123	2774.4	3415.1	5742.1	2072.9	4681.2	2901.6
AP	1995		5467.2		7131.1		6567.9	
APC (Holdford)	1872	123	1790.8	3676.4	4949.1	2182.0	3808.6	2759.3
APC (seq)	1872	123	1810.4	3656.8	4957.7	2173.4	3828.6	2739.3
AC	1890		2774.4		5742.1		4681.2	
APC (Holdford)	1872	18	1790.8	983.6	4949.1	793.0	3808.6	872.6
APC (seq)	1872	18	1810.4	964.0	4957.7	784.4	3828.6	852.6

^aover-dispersion parameter $\phi = 0.95$

^bover-dispersion parameter $\phi = 2.61$

^cover-dispersion parameter $\phi = 2.03$

Table 2: Comparison of different models for women

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